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Algebraic Formulation of Quantum Decoherence

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An algebraic formalism for quantum decoherence in systems with continuous evolution spectrum is introduced. A certain subalgebra, dense in the characteristic algebra of the system, is defined in such a way that Riemann–Lebesgue theorem can be used to explain decoherence in a well defined final pointer basis.

KEY WORDS: algebraic formulation; quantum decoherence.

1. INTRODUCTION

Quantum decoherence was a subject of intense research in the last years (Giuliani *et al*., 1996; Halliwell *et al*., 1994; Paz and Zurek, 1993, 2000; Zurek, 2001). We have contributed to this research in paper (Castagnino and Laura, 2000). Namely, we have presented an easy approach to explain decoherence in a well defined *final pointer basis*, for quantum systems with continuous evolution spectrum using a *functional* method based on a van Hove's (1955, 1956, 1957, 1959) idea.[4] We have also reobtained all the results of the method of *decoherence of histories* in the final pointer basis and defined a final intrinsic consistent set of histories (Gell-mann and Hartle, 1994; Griffith, 1984; Omnés, 1988, 1990, 1992). But our previous approach looks like an "ad hoc" one, at least in the way we presented it in papers, Castagnino and Laura (2000) and Laura and Castagnino (1998) (see also Antoniou *et al*., 1997 for a more rigorous but still "ad hoc" formalisms).

Here we will show the general nature of this approach, introducing, *a rigorous mathematical formalism for our method in the framework of the algebraic theory*

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[4] Our approach seems so easy because we directly use the eigenbasis of the perturbed hamiltonian. Namely, we bypass the problem of the diagonalization of this hamiltonian, because we are trying to describe the essence of decoherence, and this diagonalization is clearly an independent problem. Some physical examples of the method can be found in Castagnino *et al*. (2000).

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of quantum systems, which is based in a *characteristic algebra* but not in a Hilbert space representation. The importance of this "algebraic"[5] presentation lies in the fact that we may deal with systems with infinite degrees of freedom (like an idealization of a quantum gas, or a quantum field), where it is possible to have many nonequivalent representations of the commutation relations, instead of just one. Therefore, the choice itself of the adequate representation becomes a dynamical problem. Moreover, as we generally need bounded and unbounded observables, it is better to use a *nuclear* characteristic algebra (Iguri and Castagnino, 1999), whose generalized GNS-representations naturally introduce unbounded operators associated to some riggings of the algebra with a state-dependent Hilbert space (Bélanger and Thomas, 1990). As a consequence of the *nuclear theorem*, our observables will be given by nuclei or kernels—generalized matrices—and so we get a "kernel (or nucleus) mechanics" quite similar to the original "matrix mechanics."

The physical idea behind the formalism is the following: it is well known in the literature, that, in order to obtain decoherence, something like a "coarse-graining" seems to be necessary. More precisely, what produces decoherence is a combination of certain dynamical qualities of the system itself, together with an unavoidable restriction of the accessible information (as it happens in a measurement, for example). The literature contains a large list of examples studying these phenomena, namely a great set of *particular cases normally solved by numerical methods* (that include the solution of the hamiltonian diagonalization problem). Therefore, these examples can be considered even more particular than the treatment in our papers (Castagnino and Laura, 2000; Laura and Castagnino, 1998). If fact, even if extremely interesting and valuable, *they only deal with an approximate diagonalization of the density matrix of the system in a finite time*. The common features of all these examples can be heuristically explained in the following way: It is well known that the evolution of a quantum state $\rho(t)$ *does not* yield decoherence. But, if there is a projector P giving its "relevant part" $P\rho(t) = \rho_{\text{rel}}(t)$, this projected part *would* eventually decohere. In order to relate this projection to a measurement physical process let us consider the case where:

$$P = \int_{\Sigma} |O_\sigma)(O_\sigma| \, d\sigma \tag{1}$$

where $\{|O_\sigma) : \sigma \in \sum\}$ is a generalized "orthonormal" set of observables. This would be the simplest mathematical version of decoherence: that produced by a projector P related to the measurement processes that yield the mean values $(O_\sigma|\rho(t))$. *The aim of this paper is to mathematically characterize the space (or algebra) containing the observables $|O_\sigma)$ that can yield decoherence, as well as a mathematical process modelling decoherence*. So, in some way, we have to formalize the physical fact that, very frequently, we do not get *the whole* relevant

[5] It would be more precise to say "topological–algebraic" formalism.

information about the system, but only *a part* of it. Translating this into algebraic terms—in order to begin our mathematical treatment of the subject, we could say that sometimes it is impossible to use neither *all* the characteristic algebra $\mathcal{A}$ of our system, nor *all* its corresponding symmetric elements or observables, but just a certain *subalgebra* and its subset of observables (e.g. the subalgebra of the observables $|O_\sigma)$ of Eq. (1). Of course, in principle there are infinite ways of making the choice of such a subalgebra. But the essential idea is to do it in such a way that it could give us the possibility of the annihilation—at least for $t \to +\infty$—of all the "off-diagonal terms" of the states by an application of the Riemann–Lebesgue theorm.

In our subalgebra, that will be called *the van Hove algebra* $\mathcal{A}_{\mathrm{vH}}$, we will necessarily have two kinds of observables:

(i) observables a measuring the "diagonal terms," that commute with the Hamiltonian and can be given by a "singular" or "semiregular" kernel of the form: $a(\omega)\delta(\omega - \omega')$ where $a(\omega)$ is a regular function[6] Our algebra must contain this kind of observables in order to be able to define complete sets of commuting observables containing the hamiltonian.

(ii) but our subalgebra must also contain the observables that do not commute with the hamiltonian. Let us call $a(\omega)\delta(\omega - \omega') + a_r(\omega, \omega')$ the corresponding kernels. If $a_r(\omega, \omega')$ would be a generic kernel obviously there would be no loss of generality, but in this case we would retain all the algebra $\mathcal{A}$ and there would not be neither loss of information, and therefore no decoherence. So we must reduce somehow the information restraining the algebra to a smaller set. The weakest way is to make $a_r(\omega, \omega')$ a regular function, (precisely an "ordinary" functions—with respect to the two energy variables ω, ω'-, whose mathematical properties would be sufficient in order to use the Riemann–Lebesgue theorem, in its two variables case, in such a way that the off-diagonal terms $a_r(\omega, \omega')$ vanish when $t \to +\infty$). In this way we have restricted the algebra $\mathcal{A}$ to a smaller one $\mathcal{A}_{\mathrm{vH}} \subset \mathcal{A}$ with the corresponding lost of information. But as any singular kernel can be approximated by a regular one, the van Hove algebra $\mathcal{A}_{\mathrm{vH}}$ will be dense in $\mathcal{A}$. Therefore, $\mathcal{A}_{\mathrm{vH}}$ will not be *mathematically* complete, but it will be *physically* "complete," in the sense of having all the required physical properties up to any order, and therefore being *experimentally indistinguishable* from $\mathcal{A}$.

In this way the adoption of $\mathcal{A}_{\mathrm{vH}}$, represents the minimal possible coarse-graining, *i.e.* a dense coarse graining.

[6] Where ω is the energy, i.e. an eigenvalue of the hamiltonian generating the evolution, and $a(\omega)$ the regular function that represent the components of the diagonal of the operators that commute with the hamiltonian.

The paper is organized as: In Section 2, we review some basic and remarkable facts about the nuclear *-formalism for quantum mechanics. In Section 3, we introduce the nuclear *-algebra $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$, associated with the Schwartz space $\mathcal{S}(\mathbb{R}^+)$. In section 4, we first introduce a simple but very illustrative example: a quantum system whose characteristic algebra is $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$, and where the CSCO (complete set of commuting observables) is just the hamiltonian H. We define its van Hove algebra, and show how the evolution of the quantum system yields decoherence in it. In Section 5, we consider a more general quantum system, with a CSCO given by $\{H, O_1, O_2, \ldots, O_N\}$ whose spectrum is $\Lambda \subset \mathbb{R}^{N+1}$. We show how to obtain its characteristic nuclear *-algebra $\mathcal{L}(\mathcal{S}_\Lambda(\mathbb{R}^{N+1}))$ and we generalize the van Hove algebra, yielding decoherence.

Finally, in Section 6, we draw our main conclusions.

2. QUANTUM MECHANICS IN A NUCLEAR *-ALGEBRA $\mathcal{A}$

2.1. Dynamics of $\mathcal{A}$

Let us consider a nuclear *-algebra $\mathcal{A}$ (Borchers, 1972; Iguri and Castagnino, 1999; (Pietsch, 1972, chap. 3–7; Treves, 1967, part III, §47–§51) as the *characteristic algebra* of a quantum system. This amounts to say that we can describe any physical property of the system in terms of mathematical objects related to $\mathcal{A}$. *So the observables in $\mathcal{A}$ give all the information auailable*. For example, all the commutation relations of the observables[7] of the system can be expressed or represented in terms of the commutators of the algebra: $[a, b] := ab - ba$, all the physical symmetries can be represented as inner automorphism groups of the algebra, etc.[8] In particular, there exists a one parameter group of unitary inner automorphism

$$\mathcal{U}_t : \mathcal{A} \to \mathcal{A} / \mathcal{U}_t(a) = u_t a u_t^{-1} \tag{2}$$

with u_t *unitary* (i.e. : $u_t^* = u_t^{-1}$), representing the temporal evolution of the system. Usually $u_t = e^{-iHt}$ where H is the hamiltonian operator, *which will be supposed to have an absolutely continuous spectrum in a real interval contained in* $[0, +\infty)$. This hypothesis is crucial in order to use the Riemann–Lebesgue theorem. The spectrum could also contain an eigenvalue, corresponding to a bounded state (e.g., a ground state (Castagnino and Laura, 2000)).

Remark Here the concept of spectrum can be either the usual one—if we are dealing with a finite degree of freedom theory with a fixed Hilbert space—or its

[7] These observalbles are bounded or unbounded, instead of only bounded ones, as it happens within the B*-algebras. Of course, this as well as the presence of the Dirac's deltas, force us to go beyond normed spaces and to use nuclear spaces.

[8] We must remark that the correspondence between systems and characteristic algebras is not generally injective: there could be unequivalent physical systems with the same algebra. (This is not a surprise, because the same happens with the Hilbert space formalism).

generalization to nuclear *-algebras. In fact, it will be an essential hypothesis for any nuclear *-algebra in order to be physically admissible as the characteristic algebra of a quantum system, to have the "right" spectral properties. Namely, if it is the algebra of a system having a well known Hilbert space representation, its symmetric elements must have real spectrum identical to its corresponding Hilbert space representatives, etc. In particular, it can be a *nuclear b*-algebra*, that is to say a projective limit of B*-algebras, as in Iguri and Castagnino (1999), but it could be more general. $\mathcal{U}_t$ preserves the * operation: if $a^* = a$ then

$$\left(u_t a u_t^{-1}\right)^* = u_t a u_t^{-1} \tag{3}$$

In other words, the * operation is also an automorphism in $\mathcal{A}_S$, the real space of *symmetric* operators of $\mathcal{A}$, that is to say, of all $a \in \mathcal{A}$ such that $a^* = a$.

2.2. Observables in $\mathcal{A}$

$\mathcal{A}_S$ is colsed in $\mathcal{A}$, and therefore it is a *real* nuclear space that will be called the *Space of observables* of the system. If $\mathcal{A}$ is not commutative, the product of two symmetric elements will not be symmetric, and therefore $\mathcal{A}_S$ will not be a subalgebra of $\mathcal{A}$.

Let us consider a CSCO of the system. Then, there exists a unique minimal closed abelian subalgebra with unity containing it, called *the abelian subalgebra generated* by this CSCO, that will be labelled $\widehat{\mathcal{A}}$. Obviously, for the *real* part of it—i.e. with real scalars, instead of complex ones—$\widehat{\mathcal{A}_S}$ we have the relations:

$$\widehat{\mathcal{A}_S} \subset \mathcal{A}_S \subset \mathcal{A} \tag{4}$$

$\widehat{\mathcal{A}}$ is a complete subspace and a subalgebra of $\mathcal{A}$. As we are focused in decoherence, that is not a "covariant" subject,[9] all the CSCO that will be considered here contain the hamiltonian H.

We can also define the *cone of positive observables* $a \in \mathcal{A}_S$ as:

$$\begin{aligned} \mathcal{A}_{S+} &= \{a \in \mathcal{A} / \exists\, b \in \mathcal{A} : a = b^* b\} \\ &= \{a \in \mathcal{A} / \exists\, b \in \mathcal{A}_S : a = b^2\} \end{aligned} \tag{5}$$

This set is a cone since any linear combination of its elements with positive coefficients belongs to $\mathcal{A}_{S+}$.

2.3. Convex of States

The space of states is:

$$N(\mathcal{A}'_{S+}) = \{\rho / \rho \in \mathcal{A}'_S,\, \rho \geq 0,\, \rho(\mathbb{I}) = 1\} \tag{6}$$

[9] In the sense that we must have a privileged observable, namely the hamiltonian H. Also the final pointer basis will depend on H and the initial conditions (see Castagnino and Laura, 2000).

Precisely: $\rho \geq 0$ means $\rho(a) \geq 0$ for all a $a \in \mathcal{A}_{S+}$, or equivalently $\rho(b^*b) \geq 0$ for all $b \in \mathcal{A}$. As in Castagnino and Laura (2000), the generalization of the trace is $Tr(\rho) := \rho(\mathbb{I})$ where $\mathbb{I}$ is the identity operator of algebra $\mathcal{A}$. So, $\rho(\mathbb{I}) = 1$ is the normalization (or unit trace) condition. Clearly, $N(\mathcal{A}'_{S+})$ is a convex subset of $\mathcal{A}'_S$ and therefore inherited its topology. *This is the weak topology induced by the dual space $\mathcal{A}'_S$ that we discuss in Appendix A*. The particular states that are *extremals* of this convex constitute the subset of *pure states*. The finite convex combinations of pure states are called *mixed states*. The remaining states (functionals that cannot be represented in these two ways, but are expressible as *integrals* or "infinite combinations" of pure states, by a generalization of the Krein–Milman theorem (Hegerfeldt, 1975; Iguri and Castagnino, 1999; (Naimark, 1964, book, chapter I, §3, Theorem 2, page 63) are called *generalized states* (Laura and Castagnino, 1998).

There also is a trace preserving group of automorphism in the state space $N(\mathcal{A}'_{S+})$. In fact, the evolution automorphism over the observables of Eq. (2), induces the following group in the dual space of the algebra: $\mathcal{U}'_t : \mathcal{A}' \to \mathcal{A}'$ defined as:

$$(\mathcal{U}'_t \rho)(a) := \rho[\mathcal{U}_t(a)] = \rho\left(u_t a u_t^{-1}\right) \tag{7}$$

in another notation

$$(\mathcal{U}'_t \rho)(a) = \rho_t(a) = \rho[\mathcal{U}_t(a)] = \rho(a_t) \tag{8}$$

This evolution preserves the trace: $\rho_t(\mathbb{I}) = 1$ and the energy $\rho_t(H) =$ constant. Therefore it corresponds to the *Schrödinger picture*, while (2) corresponds to the *Heisenberg picture*.

Since $\widehat{\mathcal{A}}$ is a subalgebra of $\mathcal{A}$ we can consider the states corresponding to $\widehat{\mathcal{A}}$, namely the set of positive normalized functionals $N(\widehat{\mathcal{A}}'_+)$ that will be called the convex of *reduced states* with respect to $\widehat{\mathcal{A}}$. By a classical theorem of M. G. Krein ((Naimark, 1964) p. 63, Theorem 2), and the fact that the identity is an internal element of any CSCO, any reduced state can be extended to the whole $N(\mathcal{A}'_{S+})$.

Since $\widehat{\mathcal{A}}$ is commutative. its elements can just be considered as $\mathcal{A}$-valued functions of the observables of the corresponding CSCO, and its states—belonging to $N(\widehat{\mathcal{A}}'_+)$—as $\mathbb{C}$-valued functions of them. Therefore in some way they are "quasi-classical." In fact, the final state of a quantum measurement of the observables in the CSCO will be a corresponding reduced state (all the information content of the state concerning other noncommuting observables being eventually lost). So, we can describe the process of decoherence as a kind of temporal "homotopy" h_t mapping of $N(\mathcal{A}'_{S+})$ into $N(\widehat{\mathcal{A}}'_+)$ as $t \to +\propto$. At this moment we want to remark that the set of pure states of the system is defined as a particular component of the structure of the algebra, and therefore it cannot be altered by any unitary inner automorphism, including the temporal evolution of the system. In other words, the

temporal "homotopy" h_t must preserve "the boundary"—the pure states—of the convex of states. So, the evolution cannot possible act as a process of decoherence *on the whole* convex of states. Nevertheless, as we said in the introduction, this can happen *in a subset* of the mixed or generalized states: precisely in the *van Hove (or decoherent) states*, that will be defined in the following sections.

3. THE NUCLEAR *-ALGEBRA $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$

3.1. The Nuclear Algebra $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$

Let $\mathcal{S}(\mathbb{R}^+)$ be the closed subspace of the Schwarz space $\mathcal{S}(\mathbb{R})$ (Rudin, 1979, chap. 7, §3) consisting of all $C^\infty[0, +\infty]$ functions f such that:

$$p_{n,m}(f) = \sup_{x\in[0,+\infty)} (1 + x^2)^n |D^m f(x)| < \infty \tag{9}$$

i.e. the functions f and all its derivatives go to zero when $x \to +\infty$ faster than the inverse of any polynomial function. The topology is defined by the $p_{n,m}(f)$, in the sense that $f_i \to f$ if $\forall n, m : p_{n,m}(f_i \to f) \to 0$ as $i \to +\infty$. This is a closed subspace of $\mathcal{S}(\mathbb{R})$ and hence a complete metrizable nuclear space in itself. In the space of distribution or functionals $\mathcal{S}'(\mathbb{R}^+)$ we consider the strong topology, that is to say, the locally convex topology obtained from the systems of seminorms

$$p'_B(\alpha) = \sup\{|\alpha(g)|/g \in B\} \tag{10}$$

for any fundamental system of bounded subsets B of $\mathcal{S}(\mathbb{R}^+)$. Then $\mathcal{S}'(\mathbb{R}^+)$ is a nuclear space because $S(\mathbb{R}^+)$ is nuclear and metrizable (Pietsch, 1972; Treves, 1967).

Analogously, we can define $\mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^+)$ and its strong dual $\mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^+)$, constituted by *kernels distributions*, i.e., distributions in two (here nonnegative) variables.

Now, it is known (see formula (51.7) in Treves, 1967) that

$$\mathcal{S}'(\mathbb{R}^+)\widehat{\otimes}\mathcal{S}'(\mathbb{R}^+) \cong \mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^+) \cong \mathcal{L}(\mathcal{S}(\mathbb{R}^+), \mathcal{S}'(\mathbb{R}^+)) \tag{11}$$

where $\widehat{\otimes}$ denotes the completion of the tensor product (carrying its projective π-topology, or its equicontinuous ϵ-topology, because here they are equivalent). The last of these two isomorphisms is defined as follows. If $K \in S'(\mathbb{R}^+ \times \mathbb{R}^+)$, to any $f \in \mathcal{S}(\mathbb{R}^+)$, we can associate $\alpha \in \mathcal{S}'(\mathbb{R}^+)$ such that

$$\forall g \in \mathcal{S}(\mathbb{R}^+) : \alpha(g) = K(g \otimes f) \tag{12}$$

It is traditional to write it as

$$K(x, x') \leftrightarrow \alpha : [\alpha(f)](x) = \int K(x, x') f(x')\, dx' \tag{13}$$

Then, the isomorphism is given by the linear map

$$K \leftrightarrow a : a(f) = \alpha \tag{14}$$

Taking into account the relations

$$\mathcal{S}(\mathbb{R}^+) \hookrightarrow L^2(\mathbb{R}^+) \hookrightarrow S'(\mathbb{R}^+) \tag{15}$$

defined by the injections

$$f \hookrightarrow [f] \hookrightarrow \alpha_f(g) := \int f(x)g(x)\,dx \tag{16}$$

where $[f]$ is the class of functions that are a.e. (almost everywhere) equal to f, and taking into account Treves (1967, pp. 532–533), we will say that

(i) the kernel K or its associated map a are *semiregular in* x if a maps $\mathcal{S}(\mathbb{R}^+_x)$ into $\mathcal{S}(\mathbb{R}^+_{x'})$, and not only into $\mathcal{S}'(\mathbb{R}^+_{x'})$.
(ii) the kernel K or its associated map a are *semiregular in* x' if the transpose a' of a maps $\mathcal{S}(\mathbb{R}^+_x)$ into $\mathcal{S}(\mathbb{R}^+_{x'})$, and not only into $\mathcal{S}'(\mathbb{R}^+_{x'})$.
(iii) K is a *regular kernel* if it is the regular distribution given by a function $K(x, x')$ of $\mathcal{S}(\mathbb{R}^+_x \times \mathbb{R}^+_{x'})$.

For example, the Dirac's delta $\delta(x - x')$ is semiregular in both x and x', because it is symmetric in x and x', but obviously it is not a regular kernel. So, S' $(\mathbb{R}^+ \times \mathbb{R}^+)$ has a lot of physically important kernels. In fact it has *too many kernels*, to the extent of not being an algebra because of the well known product problem of the distributions. In order to avoid this obstacle, we restrict ourselves to

$$\mathcal{S}'(\mathbb{R}^+)\widehat{\otimes}\mathcal{S}(\mathbb{R}^+) \subset \mathcal{S}'(\mathbb{R}^+)\widehat{\otimes}\mathcal{S}'(\mathbb{R}^+) \cong \mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^+) \tag{17}$$

This amounts the restriction to the algebra:

$$\mathcal{A} = \mathcal{L}(\mathcal{S}(\mathbb{R}^+)) = \mathcal{L}(\mathcal{S}(\mathbb{R}^+), \mathcal{S}(\mathbb{R}^+)) \subset \mathcal{L}(\mathcal{S}(\mathbb{R}^+), \mathcal{S}'(\mathbb{R}^+)) \tag{18}$$

where the continuity is defined in the sense of $\mathcal{S}(\mathbb{R}^+)$. In fact,

$$\mathcal{L}(\mathcal{S}(\mathbb{R}^+)) \cong \mathcal{S}'(\mathbb{R}^+)\widehat{\otimes}\mathcal{S}(\mathbb{R}^+) \subset \mathcal{S}'(\mathbb{R}^+)\widehat{\otimes}\mathcal{S}'(\mathbb{R}^+) \tag{19}$$

In particular, $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ is a nuclear algebra. Thus, any element of $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ can be considered as a "generalized matrix" with a lower index corresponding to $\mathcal{S}'(\mathbb{R}^+)$ and an upper one corresponding to $\mathcal{S}(\mathbb{R}^+)$.

So, we realize the clearest and most intuitive idea of nuclearity, based in the *nuclear theorem*, and the very etymology of "nuclear algebras": they are algebras of nuclei or kernels that are multiplied as generalized matrices (Pietsch, 1972, Theorem 7.4.3, p. 115; Treves, 1967, Part III, §47–§51). In fact, let us define a linear and continuous mapping:

$$B : \mathcal{S}'(\mathbb{R}^+) \times \mathcal{S}(\mathbb{R}^+) \to \mathbb{C} \tag{20}$$

As any continuous function from a nuclear space into a Banach space is nuclear, this mapping is nuclear, and according to the nuclear theorem, there exists a linear and continuous mapping:

$$K : \mathcal{S}'(\mathbb{R}^+) \otimes \mathcal{S}(\mathbb{R}^+) \to \mathbb{C} \tag{21}$$

such that

$$B(\alpha, g) = K(\alpha \otimes g), \qquad \forall \alpha \in \mathcal{S}'(\mathbb{R}^+), \forall g \in \mathcal{S}(\mathbb{R}^+) \tag{22}$$

Now let $a \in \mathcal{A} = \mathcal{L}(\mathcal{S}(\mathbb{R}^+))$. For any $\alpha \in \mathcal{S}'(\mathbb{R}^+)$ and any $g \in \mathcal{S}(\mathbb{R}^+)$, we can define a bilinear and continuous mapping B_a:

$$B_a(\alpha, g) := \alpha[a(g)] \tag{23}$$

and since $\mathcal{A} = \mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ is a nuclear algebra $\exists\, K_a$ such that:

$$K_a(\alpha \otimes g) = \alpha[a(g)] \tag{24}$$

or as the physicists usually say:

$$K_a(\alpha \otimes g) = \int\int \alpha(x) K_a(x, x') g(x')\, dx\, dx' \tag{25}$$

Examples. It is easy to verify that:

1. If $a = \mathbb{I}$ then $K_a(\alpha \otimes g) = a(g)$ (in finite dimension it would be the contraction, whose matrix is the Kronecker delta), and $K_a(x, x') = \delta(x - x')$.
2. If $[a(g)](x) := x g(x)$, then $K_a(x, x') = x\delta(x - x')$ is semiregular in x and x'. More generally, when $[a(g)](x) := f(x)g(x)$, then $K_a(x, x') = f(x)\delta(x - x')$ is semiregular in x and x'.
3. If $K(x, x')$ is a regular kernel, for any $g(x) \in \mathcal{S}(\mathbb{R}^+)$ the function

$$f(x) := \int K(x, x') g(x')\, dx' \tag{26}$$

belongs to $\mathcal{S}(\mathbb{R}^+)$, and therefore we can define an operator $a \in \mathcal{L}(\mathcal{S}(\mathbb{R}^+))$, by:

$$[a(g)](x) := f(x) \tag{27}$$

4. If $K(x, x')$ is a general distribution, then Eq. (26) defines a tempered distribution (because in that case $f(x)$ is not necessarily a Schwarz function, moreover, it may not even be a function). Thus, we return to the correspondence

$$K(x, x') \mapsto \alpha_x : [\alpha_x(g(x'))](x) := \int K(x, x') g(x')\, dx' \tag{28}$$

which (as it was already shown), is a nonsurjective injection

$$(\mathcal{S}(\mathbb{R}^+) \otimes \mathcal{S}(\mathbb{R}^+))' \to \mathcal{L}(\mathcal{S}(\mathbb{R}^+), \mathcal{S}'(\mathbb{R}^+)) \tag{29}$$

3.2. The Star Operation in $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$

We will show that $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ also is a *-algebra. We know that for $\forall a \in \mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ there exists the *dual or transpose map* $a' : \mathcal{S}'(\mathbb{R}^+) \to \mathcal{S}'(\mathbb{R}^+)$ such that:

$$[a'(\alpha)](g) := \alpha[a(g)] \tag{30}$$

Similarly we can define $a^\dagger : \mathcal{S}'(\mathbb{R}^+) \to \mathcal{S}'(\mathbb{R}^+)$ such that:

$$[a^\dagger(\alpha)](g) := \alpha[\overline{a(\bar{g})}] \tag{31}$$

Let us define:

$$a^* = a^\dagger|_{\mathcal{S}(\mathbb{R}^+)} \tag{32}$$

with the restriction according to Eq. (16). Then

$$[\,\overline{a(\bar{g})}\,](x) = \int \overline{K_a(x, x')} g(x')\, dx' \tag{33}$$

and

$$[a^*(f)](x) = [a^\dagger(\alpha_f)](x) = \int f(x') K_{a^*}(x', x)\, dx' \tag{34}$$

But according to Eq. (31), we have

$$\begin{aligned}
[a^\dagger(\alpha_f)](g) &= \int [a^\dagger(\alpha_f)](x) g(x)\, dx \\
&= \int \int f(x') K_{a^*}(x', x) g(x)\, dx'\, dx \\
&= \int \overline{[a(\bar{f})]}(x) g(x)\, dx \\
&= \int \int f(x') \overline{K_a(x, x')} g(x)\, dx'\, dx
\end{aligned} \tag{35}$$

Thus

$$K_{a^*}(x', x) = \overline{K_a(x, x')} \tag{36}$$

and therefore the star operation is the conjugation followed by the transposition as in the case for ordinary matrices.

By its definition, it is clear that this * operation is a continuous antihomomorphism of algebras. And as $\mathcal{S}(\mathbb{R}^+)$ is a reflexive space (because it is a Montel space (Treves, 1967, p. 376), it is involutive.

Summarizing, $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ is a complete nuclear *-algebra.

4. QUANTUM MECHANICS IN $\mathcal{A} = \mathcal{L}(\mathcal{S}(\mathbb{R}^+))$

4.1. The Simplest Example

For the sake of simplicity, let us consider a physical system whose Hamiltonian has $[0, +\infty]$ as absolutely continuous spectrum, and such that $\{H\}$ is a CSCO generating $\widehat{\mathcal{A}}$ (we will generalize this CSCO in the next section). Clearly,

$$\text{Clausure in } \mathcal{A} \text{ of } \{\widehat{a} \in \mathcal{A}/K_{\widehat{a}}(\omega, \omega') = \widehat{a}(\omega)\delta(\omega - \omega'), \widehat{a}(\omega) \in \mathbb{C}[\omega]\} \subset \widehat{\mathcal{A}}$$

($\mathbb{C}[\omega]$ = set of all polynomial functions in ω with complex coefficients). As the left hand side of this inclusion relation *already is* a closed commutative subalgebra with unit of $\mathcal{A}^*$ containing $\{H\}$. the inclusion above must be an equality.

Then, $\mathcal{A} = \mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ is its natural characteristic algebra, and according to the previous section (example 2), the particular semiregular kernels of type $\widehat{a}(\omega)\delta(\omega - \omega')$[10] correspond to the elements of $\hat{A}$.

4.2. The van Hove Algebra

Let us consider the quotient $\mathcal{A}/\mathcal{A}$ which is a vector space (but not a subalgebra since $\widehat{\mathcal{A}}$ is not an ideal). Let us call

$$\mathcal{A}/\widehat{\mathcal{A}} := \mathcal{V}_{\text{nd}} \tag{37}$$

the "nondiagonal" vector space. Then, if $[a] \in \mathcal{V}_{\text{nd}}$ and $a \in \mathcal{A}$:

$$[a] = \widehat{\mathcal{A}} + a \tag{38}$$

so

$$\mathcal{A} = \widehat{\mathcal{A}} + \mathcal{V}_{\text{nd}} \tag{39}$$

where the last "+" symbol is not a direct sum, since we can add and substract an arbitrary $a \in \widehat{\mathcal{A}}$ to each term of the r.h.s. But we can turn "+" into a "$\oplus$" if we restrict ourselves to a smaller (but dense, and so physically equivalent) subalgebra of $\mathcal{A}$.

In general, the kernels of $\mathcal{V}_{\text{nd}}$ are tempered distributions. Now, let us restrict these last kernels to be just regular ones, constituting a space $\mathcal{V}_r \subset \mathcal{V}_{\text{nd}}$. Then we can define the *van Hove algebra* as:

$$\begin{aligned} \mathcal{A}_{\text{vH}} &= \widehat{\mathcal{A}} \oplus \mathcal{V}_r \\ &= \{a \in \mathcal{A}/K_a(\omega, \omega') = \widehat{a}(\omega)\delta(\omega - \omega') + a_r(\omega, \omega')\} \end{aligned} \tag{40}$$

[10] Remember that here the "$\widehat{a}$" symbol does not mean "a is an operator," as often used in quantum books, but it indicates association with $\widehat{\mathcal{A}}$.

where $\widehat{a}(\omega)$, and $a_r(\omega, \omega')$ are "regular" functions, in the sense of being endowed with the properties listed in the Appendix of paper (Laura and Castagnina, 1998) (they were chosen to be *natural* from a physical point of view, and *sufficient* in order to satisfy the mathematical hypothesis of the Riemann–Lebesgue theorem, *in the two variables case*), namely:

1. $\widehat{a}(\omega) \in \mathcal{S}(\mathbb{R}^+)$
2. $a_r(\omega, \omega') \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^+)$

Now we have a $\oplus$ because a kernel cannot be a Dirac's δ and a regular function at the same time. It is easy to prove that $\mathcal{A}_{\text{vH}}$ is a *-subalgebra of $\mathcal{A}$, and hence a nuclear algebra in itself, but a noncomplete one (because it is not closed). Nevertheless, it is dense in $\mathcal{A}$ (because as it is well known, any distribution is a limit of regular functions). Anyhow, here the noncompleteness is not a problem, because we are not interested in taking general limits in $\mathcal{A}_{\text{vH}}$. Let us denote $\widehat{a}$, $a_r \in \mathcal{A}$ the linear operators whose kernels are $\widehat{a}(\omega)\delta(\omega - \omega')$ and $a_r(\omega, \omega')$, respectively. Then we can write

$$a = \widehat{a} + a_r$$

Now consider the time evolution within the van Hove algebra. If $a \in \mathcal{A}_{\text{vH}}$ then

$$\mathcal{U}_t(a) = \mathcal{U}_t(\widehat{a} + a_r) = \widehat{a} + \mathcal{U}_t(a_r) \tag{41}$$

where $\widehat{a} \in \widehat{\mathcal{A}}$ and $a_r \in \mathcal{V}_r$. The last equation shows the most important characteristic of the semiregular and regular parts: the semiregular part $\widehat{a}$ is *invariant* under time evolution while the regular a_r is *fluctuating*.

As we are particularly interested in observables, i.e. symmetric elements of the algebra, we define the *space of van Hove observables*,

$$\mathcal{A}_{\text{vHS}} := \{a \in \mathcal{A}_{\text{vH}} / a^* = a\} \tag{42}$$

In particular, $a \in \mathcal{A}_{\text{vHS}}$ implies that:

3. $\hat{a}(\omega)$ is a *real*-valued regular function of $\mathcal{S}(\mathbb{R}^+)$, and
4. the regular term is *hermitian*, i.e., it verifies

$$a_r(\omega, \omega') = \overline{a_r(\omega', \omega)}$$

4.3. The van Hove States

Now, we are going to define the states. First, let

$$\mathcal{A}'_{\text{vH}} = \widehat{A}' \oplus \mathcal{V}'_r \tag{43}$$

where $\widehat{\mathcal{A}}'$ is the topological dual of $\widehat{\mathcal{A}}$, but $\mathcal{V}_r'$ is just a notation for the set of all functionals[11] $\rho_r \in \mathcal{A}'$, whose kernels $\rho_r(\omega, \omega')$ satisfy:

1. $\rho_r(\omega, \omega')a_r(\omega', \omega) \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^+)$ for any $a_r \in \mathcal{V}_r$.
   Then, if $a \in \mathcal{A}_{\text{vH}}$ we have:

$$\rho(a) = \int_0^{+\infty} \widehat{\rho}(\omega)\widehat{a}(\omega)d\omega + \int_0^{+\infty}\int_0^{+\infty} \rho_r(\omega, \omega')a_r(\omega', \omega)\, d\omega'\, d\omega \tag{44}$$

   If they also satisfy:
2. $\widehat{\rho}(\omega)$ is a *real*-valued regular function of $\mathcal{S}(\mathbb{R}^+)$, and
3. the regular term is *hermitian*, i.e., it verifies

$$\rho_r(\omega, \omega') = \overline{\rho_r(\omega', \omega)}$$

   they belong to

$$\mathcal{A}'_{\text{vHS}} = \widehat{\mathcal{A}}'_{\text{S}} \oplus \mathcal{V}'_{r\text{S}} \tag{45}$$

   If in addition they also satisfy:
4. $\widehat{\rho}(\omega)$ is a *positive*-valued regular function of $\mathcal{S}(\mathbb{R}^+)$, and
5. $\rho_r(\omega, \omega')$ is a *positive kernel* they belong to

$$\mathcal{A}'_{\text{vHS}+} = \widehat{\mathcal{A}}'_{\text{S}+} \oplus \mathcal{V}'_{r\text{S}+} \tag{46}$$

   Finally, if they verify:
6. the normalization condition

$$\rho(\mathbb{I}) = \int_0^{+\infty} \widehat{p}(\omega)d\omega = 1 \tag{47}$$

   then we get a van Hove state $\rho \in N\ (\mathcal{A}'_{\text{vHS}+})$:

$$N(\mathcal{A}'_{\text{vHS}+}) = N(\widehat{\mathcal{A}}'_{\text{S}+}) \oplus \mathcal{V}'_{r\text{S}+} \tag{48}$$

   (There is no normalization in $\mathcal{V}'_{r\text{S}+}$ since $\mathbb{I} \in \widehat{\mathcal{A}}_{\text{S}+}$). As before, we will write

$$\rho = \widehat{\rho} + \rho_r \tag{49}$$

   and so

$$\rho(a) = \widehat{\rho}(\widehat{a}) + \rho_r(a_r) \tag{50}$$

   Let us consider the time evolution:

$$(\mathcal{U}_t\rho)(a) = \rho(\mathcal{U}_t(a)) = \widehat{\rho}(\widehat{a}) + \rho_r(\mathcal{U}_r(a_r)) \tag{51}$$

[11] These are not *states*, because they will have null trace. But for this same property, each of them added to any reduced state $\widehat{\rho} \in N(\widehat{\mathcal{A}}'_{\text{S}+})$ will bring another state ρ of $N(\mathcal{A}'_{\text{S}+})$.

We can see that the singular part of the states is *invariant* under time evolution, while the regular one fluctuates.

Theorem 1. *Any van Hove state of* $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ *becomes weakly-star diagonal for* $t \to +\infty$

Proof: Let us compute the evolution of the regular part of a van Hove state:

$$\begin{aligned}\rho_r(\mathcal{U}_t(a_r)) &= \int\int (\mathcal{U}_t(a_r))(\omega', \omega)\rho_r(\omega, \omega')\, d\omega\, d\omega' \\ &= \int\int e^{i\omega' t} a_r(\omega', \omega) e^{-i\omega t} \rho_r(\omega, \omega')\, d\omega\, d\omega' \\ &= \int\int e^{i(\omega'-\omega)t} \rho_r(\omega, \omega') a_r(\omega', \omega)\, d\omega\, d\omega' \end{aligned} \tag{52}$$

where the region of integration is the cartesian square of an interval of the real numbers, according to the spectral hypothesis we have made.

All the functions in the integrand of the r.h.s. of Eq. (52) have being endowed with the properties listed in the Appendix of paper (Laura and Castagnino, 1998) (this is sufficient to satisfy the hypothesis of the Riemann–Lebesgue theorem, in the two variables case), and therefore we can conclude:

$$\lim_{t\to\infty} \rho_r(\mathcal{U}_t(a_r)) = 0 \tag{53}$$

So, we have proved that (Rudin, 1979, §3.14, p. 65)

$$w^* - \lim_{t\to\infty} \rho_r = 0 \Rightarrow w^* - \lim_{t\to\infty} \rho = \widehat{\rho} \tag{54}$$

or equivalently: in this simple case any van Hove state becomes weakly-star diagonal when $t \to +\infty$, as we were trying to prove. □

Remark This means decoherence in a purely algebraic framework. Let us remark that this is the relevant physical limit according to the available information, which can *only* be obtained through the observables a $a \in \mathcal{A}_{\mathrm{vH}}$. In fact, it is impossible to obtain any other information. Let us also remark that the *characteristic decoherence time*, namely the characteristic time in which the nondiagonal components $\rho_r(\mathcal{U}_t(a_r))$ vanish can be computed from the poles of the Liouville operator (see Appendix B). The characteristic time corresponds to the inverse of the imaginary part closer to the real axis (see Laura *et al.*, 1999; Laura and Castagnino, 1998). Therefore, if the characteristic time is extremely large, on experimental standards, there is only a theoretical decoherence, not an experimentally observable one (also see the study of this subject in paper (Castagnino and Laura, 2000)

5. QM IN THE NUCLEAR *-ALGEBRA $\mathcal{L}(\mathcal{S}_\Lambda(\mathbb{R}^{N+1}))$

Having established (for the simplest case, and when there is some fixed Hilbert space $\mathcal{H}$) the decoherence in the energy H we can consider its generalization for more complex systems, as well as the decoherence in other commuting constants of the motion $O_1, \ldots, O_N$ as in paper (Castagnino and Laura, 2000, section II. B). Moreover, we would like to show the generalization of the simplest case of finite discrete indices, considered in that paper, to finite but otherwise arbitrary kinds of indices.

Now, let the CSCO be $\{H, O_1, \ldots, O_N\}$, where we will suppose—as usually—that all the observables are bounded or unbounded essentially self adjoint operators (Reed and Simon, 1980, chap. VIII, p. 256) of a Hilbert states space $\mathcal{H}$ Having a common dense domain.

According to the *nuclear spectral theorem* (Bohm, 1978; Gelfand and Vilenkin, 1967), there exists a nuclear space Φ and its rigging with $\mathcal{H}$[12]:

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{55}$$

such that:

(i) all the observables of the CSCO have Φ as a common dense domain, and are elements of the algebra $\mathcal{L}(\Phi)$ of continuous linear operators from Φ to Φ.

(ii) there is a Φ-complete spectral resolution of the CSCO, in the sense that there is a basis of generalized eigenvectors $\{|\omega, o_1, \ldots, o_N\rangle\} \subset \Phi^\times$[13], and a numerical measure μ on the spectrum

$$\Lambda = \Lambda_1 \times \cdots \times \Lambda_{N+1} \subset \mathbb{R}^{N+1}$$

of the closure $\overline{H}, \overline{O_1}, \ldots, \overline{O_N}$ of the elements of the CSCO, such that

$$\forall \varphi, \psi \in \Phi : \langle \varphi, \psi \rangle = \int_\Lambda \langle \varphi | \omega, o_1, \ldots, o_N \rangle \langle \omega, o_1, \ldots, o_N | \psi \rangle \, d\mu \tag{56}$$

where in l.h.s. it is indicated the scalar product of φ and ψ in $\mathcal{H}$ (with the antilinear factor in the left), whereas in the r.h.s. $\langle \varphi | \omega, o_1, \ldots, o_N \rangle$ means the (antilinear) generalized right eigenvector evaluated in φ and $\langle \omega, o_1, \ldots, o_N | \psi \rangle$ the (linear) generalized left eigenvector evaluated in ψ. If it happens that some of the O_j have purely point spectra, then the factors of measure μ on the Λ_j will be atomic and the corresponding

[12] From now on $\Phi^\times$ will be the *antidual* of the nuclear space Φ i.e., the space of all continuous *antilinear* functionals on Φ, whereas its *dual*, i.e., the space of all continuous *linear* functionals, will be Φ'.

[13] $|\omega, 0_1, \ldots, o_N\rangle$ is the ket $|\omega, m_1, \ldots m_N\rangle$ of paper (Castagnino and Laura, 2000).

integrals in Eq. (56) are really sums. So, this is a generalization of the pure point spectrum case considered in (Castagnino and Laura (2000).

Denoting

$$\Omega = (\omega, o_1, \ldots, o_N) \in \Lambda \tag{57}$$

$$\Omega' = (\omega', o'_1, \ldots, o'_N) \in \Lambda \tag{58}$$

it is possible to simplify the forthcoming notation. For example, Eq. (56), now is:

$$\forall \varphi, \psi \in \Phi : \langle \varphi, \psi \rangle = \int_\Lambda \langle \varphi | \Omega \rangle \langle \Omega | \psi \rangle \, d\mu \tag{59}$$

Let us consider the set $S_\Lambda(\mathbb{R}^{N+1})$ formed by the restrictions to Λ (the spectrum of the CSCO) of all the functions belonging to the Schwarz space $\mathcal{S}(\mathbb{R}^{N+1})$. Being the image of the natural "onto" linear map Π

$$f \mapsto \Pi(f) = f|_\Lambda$$

$\mathcal{S}_\Lambda(\mathbb{R}^{N+1})$ can be considered as a quotient space

$$\mathcal{S}_\Lambda(\mathbb{R}^{N+1}) \cong \mathcal{S}(\mathbb{R}^{N+1})/Ker(\Pi)$$

of the nuclear space $\mathcal{S}(\mathbb{R}^{N+1})$ modulo the closed linear subspace $Ker(\Pi)$. By a well-known result (Treves, 1967, Part III, §50, Proposition 51, page 514) this space is nuclear. There are good physical reasons for such a choice. In fact, any physical observable of a quantum system like those considered here, decreases fast or even vanishes at infinity, and it only matters within the limits of its own spectrum, where all the results of its experimental measures lay.

Now, any observable or "generalized matrix" of our system must be a kernel $O(\Omega, \Omega')$, and the product must be a generalization of the product of matrices, such as the composition of linear mappings into infinite-dimensional vector spaces. Taking these ideas in mind, as well as the experience left by our previous example, let us consider as the characteristic algebra

$$\mathcal{A} = \mathcal{L}(\mathcal{S}_\Lambda(\mathbb{R}^{N+1})) \cong \mathcal{S}'_\Lambda(\mathbb{R}^{N+1}) \widetilde{\otimes} \mathcal{S}_\Lambda(\mathbb{R}^{N+1}) \tag{60}$$

By the same arguments as before, $\mathcal{A}$ is a nucleare *-algebra. Moreover, it is dual-nuclear, i.e., its dual $\mathcal{A}'$ is also nuclear. In fact

$$\mathcal{A}' \cong \mathcal{S}_\Lambda(\mathbb{R}^{N+1}) \widehat{\otimes} \mathcal{S}'_\Lambda(\mathbb{R}^{N+1}) \tag{61}$$

As before, let $\widehat{\mathcal{A}}$ be abelian subalgebra generated by this CSCO, and define:

$$\begin{aligned} \mathcal{A}_{\mathrm{vH}} &= \widehat{\mathcal{A}} \oplus \mathcal{V}_r \\ &= \{a \in \mathcal{A} / K_a(\Omega, \Omega') = \widehat{a}(\Omega)\delta(\omega - \omega') + a_r(\Omega, \omega')\} \end{aligned} \tag{62}$$

where $\widehat{a}(\Omega)$, $a_r(\Omega, \Omega')$ are complex-valued regular *functions*. If we only consider symmetric operators, we add the subindex "$\mathcal{S}$," getting

$$\mathcal{A}_{\mathrm{vHS}} = \widehat{\mathcal{A}}_{\mathrm{S}} \oplus \mathcal{V}_{r\mathrm{S}} \tag{63}$$

With regard to functionals, we demand

$$\begin{aligned}\mathcal{A}'_{\mathrm{vHS}+} &= \widehat{A}'_{\mathrm{S}+} \oplus \mathcal{V}'_{r\mathrm{S}+} \\ &= \{\rho \in \mathcal{A}'/K_\rho(\Omega, \Omega') = \widehat{\rho}(\Omega)\delta(\omega - \omega') + \rho_r(\Omega, \Omega')\}\end{aligned} \tag{64}$$

where $\widehat{\rho}(\Omega)$ is a *positive*-valued regular *function*, and $\rho_r(\Omega, \Omega')$ is now a *positive kernel* verifying: $\rho_r(\Omega, \Omega')a_r(\Omega', \Omega) \in \mathcal{S}(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1})$ for any $a_r \in \mathcal{V}_r$.

As before, adding the normalization condition

$$\rho(\mathbb{I}) = \int \cdots \int_\Lambda \widehat{\rho}(\Omega)\, d\mu = 1 \tag{65}$$

we obtain the *van Hove states*

$$N(\mathcal{A}'_{\mathrm{vHS}+}) = N(\widehat{A}'_{\mathrm{S}+}) \oplus \mathcal{V}'_{r\mathrm{S}+} \tag{66}$$

If $a \in \mathcal{A}_{\mathrm{vH}}$ we have:

$$\rho(a) = \widehat{\rho}(\widehat{a}) + \rho_r(a_r) \tag{67}$$

Theorem 2. *Any van Have state of* $\mathcal{L}(\mathcal{S}_\Lambda(\mathbb{R}^{N+1}))$ *becomes weakly-star diagonal when* $t \to +\infty$

Proof: As we did in Theorem 1, let us compute the evolution of the regular part of a van Hove state of our algebra, with the help of the Riemann–Lebesgue theorem for the 2-variables case[14]:

$$\lim_{t\to\infty} \rho_r(\mathcal{U}_t(a_r)) = \lim_{t\to\infty} \int \int e^{i(\omega'-\omega)t} \rho_r(\Omega, \Omega')a_r(\Omega', \Omega)\, d\omega\, d\omega' \tag{68}$$

$$= 0 \tag{69}$$

and so

$$\lim_{t\to\infty} \rho(\mathcal{U}_t(a)) = \widehat{\rho}(\widehat{a}) \tag{70}$$

Thus, we have reached the *time-independent* component $\widehat{\rho}(\widehat{a})$ of $\rho(a)$, defined by the initial conditions. Therefore, it would be impossible that another decoherence process would *follow* in order to eliminate *the off-diagonal terms of the remaining N dynamical variables that are still present* in $\widehat{\rho}(\widehat{a})$. This is because

[14] Not the $2(N+1)$ variable case, because the exponential comes from the evolution operator and thus it *only* includes the ω, ω'-variables. There are no $\exp[i(O_j - O'_j)t]$ terms for $1 \le j \le N$. But this is what matter (for $\omega \neq \omega'$!) in the generalization of Eq. (52), which just replaces the ω's by Ω's.

$\omega = \omega'$ in Eq. (70), and there is no Riemann–Lebesgue "destructive interference" term as in the integrand of Eq. (68) for the remaining N variables. Nevertheless, by the generalized GNS-representation theorem for $\widehat{\mathcal{A}}$ (Bélanger and Thomas, 1990; Iguri and Castagnino, 1999), if $\mathcal{H}_{\widehat{\rho}}$ is the "Hilbert–Liouville" space obtained by completion of $\widehat{\mathcal{A}}$ with the pre-hilbertain scalar product "$(-|-)$" defined by $\widehat{\rho}$, i.e.

$$\forall \widehat{a}, \widehat{b} \in \widehat{\mathcal{A}} : (\widehat{a}, \widehat{b}) := \widehat{\rho}(\widehat{a}^* \widehat{b}) \tag{71}$$

there is a *pointer representation*[15]:

$$\pi_{\widehat{\rho}} : \widehat{\mathcal{A}} \to \mathcal{L}(\mathcal{H}_{\widehat{\rho}})$$

namely, the left multiplication by $\widehat{a}$

$$\pi_{\widehat{\rho}}(\widehat{a})[\widehat{b}] := [\widehat{a}\widehat{b}] \tag{72}$$

where $[\widehat{b}]$ means the equivalence class correspondent to $\widehat{b}$ in the completion of $\widehat{\mathcal{A}}$, such that, for some *normal state "vector"* (in our quantum theory it *really* has the status of an operator, like a density matrix) of the representation $|\widehat{R})$, belonging to a dense subset $D_{\widehat{\rho}}$ of $\mathcal{H}_{\widehat{\rho}}$ containing $\widehat{\mathcal{A}}$, and contained in the domains of all the essentially self adjoint "super-operators" (they act on operators) $\pi_{\widehat{\rho}}(\widehat{A}_S)$, we have

$$\forall \widehat{a} \in \widehat{\mathcal{A}} : (\widehat{R}|\pi_{\widehat{\rho}}(\widehat{a})|\widehat{R}) = \widehat{\rho}(\widehat{a}) \tag{73}$$

So, we are again under the hypothesis of the nuclear spectral theorem for the *final pointer CSCO*[16]

$$\{\pi_{\widehat{\rho}}(H), \pi_{\widehat{\rho}}(O_1), \ldots, \pi_{\widehat{\rho}}(O_N)\} \tag{74}$$

of $\mathcal{H}_{\widehat{\rho}}$ and therefore there exists a nuclear space Ψ and a rigging of it with $\mathcal{H}_{\widehat{\rho}}$

$$\Psi \subset \mathcal{H}_{\widehat{\rho}} \subset \Psi^{\times} \tag{75}$$

such that:

(i) all the observables of the CSCO have Ψ as a common dense domain, and are elements of the algebra $\mathcal{L}(\Psi)$ of continuous linear operators form Ψ to Ψ.

(ii) there is a Ψ-complete spectral resolution of the CSCO, in the sense that there is a basis of generalized eigenvectors[17]

$$\left\{ |\Theta) = |\theta_1, \overset{N-1}{\ldots}, \theta_{N+1}) \,/\, (\theta_1, \overset{N-1}{\ldots}, \theta_{N+1}) \in \sum \right\} \subset \Psi^{\times} \tag{76}$$

[15] This representation depends on $\widehat{\rho}(\widehat{a})$ which, in turn, depends on H and the initial conditions. This fact is in agreement with all the literature on the subject.

[16] Actually it is a CSCO, because $\pi_{\widehat{\rho}}$ is a homomorphism of *-algebras. This CSCO was denoted by $\{H, P_1, \ldots, P_N\}$ in paper (Castagnino and Laura, 2000).

[17] This is the basis denoted by $|\omega, r_1, \ldots, r_N)$ in paper (Castagnino and Laura, 2000).

where

$$\sum = \sum\nolimits_1 \times \ldots \times \sum\nolimits_{N+1} \subset \mathbb{R}^{N+1}$$

is the spectrum of the closures $\overline{\pi_{\widehat{\rho}}(H)}$, $\overline{\pi_{\widehat{\rho}}(O_1)}$, $\ldots$, $\overline{\pi_{\widehat{\rho}}}(O_N)$, and a numerical positive measure σ on $\sum$, such that

$$\forall \widehat{a}, \widehat{b} \in \widehat{\mathcal{A}} \cap \Psi : (\widehat{a}, \widehat{b}) = \int_{\sum} (\widehat{a}|\Theta)(\Theta|\widehat{b})\, d\sigma = \widehat{\rho}(\widehat{a}^* \widehat{b}) \tag{77}$$

In particular, for $\widehat{a} = \mathbb{I}$[18] and $\widehat{b} = \widehat{a}$

$$\begin{aligned} \widehat{\rho}(\widehat{a}) &= \int_{\sum} (\mathbb{I}|\Theta)(\Theta|\widehat{a})\, d\sigma \\ &= \left\{ \int_{\sum} \widehat{\rho}(\Theta)(\Theta| d\sigma \right\} |\widehat{a}) \end{aligned} \tag{78}$$

where $\widehat{\rho}(\ominus) := (\mathbb{I}|\ominus)$.

Thus, star-weakly (i.e. when evaluating in $\widehat{a} \in \widehat{\mathcal{A}}$), we have:

$$\widehat{\rho} = \int_{\sum} \widehat{\rho}(\Theta)(\Theta|\, d\sigma \tag{79}$$

Equation (79) shows that $\widehat{\rho}$[19] is star-weakly diagonal, but this time *in all its indices*.

Then

$$\rho = \widehat{\rho} + \rho_r = \int_{\sum} \widehat{\rho}(\Theta)(\Theta| d\sigma + \rho_r \tag{80}$$

and

$$w^* - \lim_{t \to \infty} \rho = \widehat{\rho} \tag{81}$$

as we needed to prove. □

Definition. The state-dependent basis (76) in which *all* the off-diagonal components of ρ star-weakly go to zero when $t \to \infty$, is called the *final pointer basis*. In it we have complete decoherence.

Summarizing:

(i) Decoherence in the energy is produced by the time evolution when $t \to +\infty$.
(ii) Decoherence in the other dynamical variables appears if we choose an adequate generalized eigen-basis, namely the final pointer basis.

[18] It can be supposed without loss of generality that $\mathbb{I} \in \widehat{\mathcal{A}} \cap \Psi$. In fact, $\mathbb{I} \in \widehat{\mathcal{A}}$ and so, $\mathbb{I} \notin \Psi$ we add $\pi_{\widehat{\rho}}(\mathbb{I})$ to the final pointer CSCO, and it takes a new Ψ.

[19] Denoted ρ, in Castagnino and Laura (2000).

6. CONCLUSIONS

In the introduction we said that the cause of decoherence is a combination of certain dynamical properties of the system itself, together with an unavoidable restriction of the accessible information.

Then, two final comments are in order.

1. It is necessary that the system has an absolutely continuous evolution spectrum,[20] which implies a certain degree of complexity (in fact, classically mixing dynamical systems have this kind of spectra ((Reed and Simon, 1980) chap. VII, §4, p. 239). Moreover, from the use of the Riemann–Lebesgue theorem we may have the feeling that *all* systems do decohere. While this is theoretically true when $t \to \infty$, it is not so for finite time. This problem is discussed in paper (Castaguino and Laura, 2000), and it turns out that systems with an infinite characteristic time do not decohere, if considered on physical grounds. Therefore, continuous spectrum and finite decoherence characteristic time (as computed in Appendix B) are the dynamical properties needed for the system to decohere.

2. With respect to decoherence, what Physics finally has to deal with is not the whole characteristic algebra of the system, but only the *actual* set of its measurable *observables*. This is the unavoidable restriction of the accessible information. Therefore, if we choose to work with an algebraic formalism, our task would be to find some "method of restriction" of the characteristic algebra, or of its observables, in order to explain the underlying mechanism behind this kind of phenomena. Ideally, we would like this restriction to be not too strong as to lead us too far away of our original system model. On the contrary, we would like to be as close as possible to the algebra $\mathcal{A}$. That is why we have chosen an $\mathcal{A}_{vH}$ which is dense in $\mathcal{A}$.

A lot of work must be done to complete the scenario presented in this paper:

1. The physical nature of the decohered states, that we have mathematically described, and also their relation with their numerical approximations must be precised. We will develop this subject elsewhere.
2. More examples than those given in and Castagnino and Laura (2000) Catagnino *et al.* (2000) must be presented to connect our formalism to the usual physical examples (nevertheless, Castagnino *et al.*, 2002, deals with the Mott problem: the classical radial trajectories produced in the bubble chamber by radioactive nucleus is an eloquent sample of this connection).

[20] Using our method, of course. But other methods put the system into a box, and sooner or later they make a limit process sending its size to infinity. So, they just use an "approximative version" of our continuous spectrum hypothesis.

APPENDIX A. THE TOPOLOGY

Essentially, quantum mechanics is based on three axioms (Ballentine, 1998):

1. If you measure an observable you get an eigenvalue of its spectrum.
2. The mean value of an observable A in a state ρ is $T\tau(\rho A)$. That can be generalized, as we have done in this paper, saying that the mean value is the result of the functional ρ over the observable A.
3. Schroedinger's evolution equation, or its equivalents as Eqs. (2) or (7). This can be derived from other considerations (see Ballentine, 1998, Chapter III, and Jordan, 1975).

All the mathematical structures must be derived from these axioms, or must be adopted by simplicity or convenience.

In this scenario we can compare the weak star topology with the norm topology. The weak star topology is a direct consequence of axiom 2 in the limit $t \to \infty$. In fact, this axiom defines the relation among observables, states, and mean values that we have used, and yields to the weak limits, so introducing the weak star topology. The norm topology arises as a consequence of using a—traditional—Hilbert space of pure states, what can be justified by its success in doing quantum mechanics ("a la Dirac") for systems having only *bounded* operators. But in the case of quantum systems with *unbounded* operators, while the motivation for taking the weak star topology stands, the bases for the use of the norm topology trembles, at least if we want to work "a la Dirac," and not "a la Weil," i.e., exponentiating the unbounded operators (Reed and Simon, 1980, Chapter VIII). In fact, in that case we are forced to change the Hilbert space by a nuclear space (carrying the desired distributions and kernels). So, we *must* prefer the first choice. Thus, if two states coincide with respect to the weak star topology (and therefore, coincide from the point of view of the second axiom), but they are different with respect to the norm topology, we are *experimentally* induced to choose their equality, as well as to suspect of the completely *theoretical* statement that they are different. Let us give two arguments in support of this opinion.

1. It is well known that a unitary evolution never makes diagonal a nondiagonal matrix ρ. Namely, we cannot have a norm limit $\|\rho - \rho_*\| \to 0$. But what actually is possible—and is the essential point here—is the limit of the "relevant parts" of ρ and ρ_*, i.e., of its projections or traces: $\|\rho_R - \rho_{*R}\| \to 0$, where ρ_R and ρ_{*R} are defined by

$$\rho_R = \sum_{i=1}^{n} |\phi_i)(\phi_i|\rho); \qquad \rho_{*R} = \sum_{i=1}^{n} |\phi_i)(\phi_i|\rho_*)$$

being $\sum_{i=1}^{n} |\phi_i)(\phi_i|$ a generic projector.

This essential fact can be proved using our weak topology. In fact, let us consider

$$\|\rho_R - \rho_{*R}\| = \left\| \sum_{i=1}^{n} |\phi_i)(\phi_i|\rho - \rho_*) \right\| \leq \sum_{i=1}^{n} \|\phi_i\| \cdot |(\phi_i|\rho) - (\phi_i|\rho_*)|$$

$$= \sum_{i=1}^{n} |(\phi_i|\rho) - (\phi_i|\rho_*)|$$

since $\|\phi_i\| = 1$. Then if it is true that $\rho - \rho_*$ weakly, then

$$\forall i = 1, \ldots, n : \lim_{t\to\infty} |(\phi_i|\rho(t)) - (\phi_i|\rho_*)| = 0 \Rightarrow \|\rho_R(t) - \rho_{*R}\| \to 0$$

2. We cannot find the limit in norm for the single vectors of an evolving pointer basis. An example is in order (Bacciagaluppi and Hemmo, 1994). Using the concepts of standard theory (e.g. Paz and Zurek, 1993; Zurek, 2000, 2001) let us study a simple two-dimensional discrete space system coupled to an environment. The traced density matrix (see Eq. (11) of Bacciagaluppi and Hemmo (1994) reads

$$\rho_R(t) = \frac{1}{2}\begin{pmatrix} 1 & \alpha(t) \\ \alpha(t) & 1 \end{pmatrix}$$

When $t \to \infty$ we are entitled to believe that the final state is

$$\rho_{*R} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The trace norm would be

$$\|\rho_R(t) - \rho_{*R}\| = Tr[(\rho_R(t) - \rho_{*R})^\dagger(\rho_R(t) - \rho_{*R})] = \frac{1}{2}|\alpha(t)|^2$$

so if we use a trace norm limit, when $t \to \infty$, we will have $|\alpha(t)| \to 0$. But $\alpha(t) = |\alpha(t)|e^{i\omega(t)}$, and so, at time t, the eigenvalues and (normalized) eigenvectors of $\rho_R(t)$ are:

$$\frac{1}{2}(1 \pm |\alpha(t)|); \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{-i\omega(t)} \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega(t)} \\ 1 \end{pmatrix}$$

Thus, in the limit $|\alpha(t)| \to 0$, we obtain the eigenvalues of ρ_{*R} but not its eigenvectors, because *we do not know the behavior of* $\omega(t)$ *when* $t \to \infty$. Namely, the norm topology does not satisfy our hope that the eigenvectors of $\rho_R(t)$ would tend to the pointer basis (see Bacciagaluppi and Hemmo (1994) for details). This problem seems very difficult to solve. At least in paper (Bacciagaluppi and Hemmo, 1994), the authors are forced to invoke assumptions and provisos to complain with the experimental results.

On the other hand, in the weak topology, we have

$$\lim_{t\to\infty} \rho_R(t)(a) = \rho_{*R}(a), \quad \forall_a \in \mathcal{A}_S$$

and the problem disappears, because the eigenbasis of $\rho_R(t)$ at finite time t is of no importance in our formalism (since it is not used) and therefore, must not be considered. The last equation says that, for all measurements done with any measurement apparatus, we see $\rho(t) \to \rho_*$ which is clearly the actual physical experimental result.[21] So, what we have discussed here is just a *pseudoproblem*, due to the use of inadequate concepts that yields an unmotivated choice of the topology.

APPENDIX B: DECOHERENCE TIME

In this appendix we will give an example of computation of the decoherence time following and generalizing the method of paper (Laura *et al.*, 1999). Let the free hamiltonian be

$$H^0 = \sum_{j=1}^{N} \Omega_j |j\rangle\langle j| + \int_0^\infty d\omega |\omega\rangle\langle\omega| \tag{82}$$

and the interaction hamiltonian be

$$H^1 = \sum_{j=1}^{N} \int_0^\infty d\omega [V_{\omega j}|\omega\rangle\langle j| + V_{j\omega}|j\rangle\langle\omega|] \tag{83}$$

where the V's are considered small in order to introduce a perturbative process. The generic observable of space $\mathcal{A}_S$ reads

$$O = \sum_{jj'} O_{jj'}|j\rangle\langle j'| + \int d\omega O_\omega |\omega\rangle\langle\omega| + \int d\omega \int d\omega' O_{\omega\omega'}|\omega\rangle\langle\omega'|$$
$$+ \int d\omega \sum_{j'} O_{\omega j'}|\omega\rangle\langle j'| + \int d\omega \sum_{j'} O_{j'\omega}|j'\rangle\langle\omega|$$

We will follow the convention of using "round kets" for the vectors of the basis of space $\mathcal{A}_S$, built using the eigenvectors of the free hamiltonian H^0, as

$$|jj') = |j\rangle\langle j'|, |\omega) = |\omega\rangle\langle\omega|, |\omega\omega') = |\omega\rangle\langle\omega'|, |\omega j') = |\omega\rangle\langle j'|, |j\omega') = |j\rangle\langle\omega'|$$

with "round bra" namely $(jj'|$, $(\omega|$, $(\omega\omega'|$, $(\omega j'|$, $(j\omega'|$, the corresponding generalized cobasis, and

$$(\Psi_{jj'}|, (\Psi_\omega|, (\Psi_{\omega\omega'}|, (\Psi_{\omega j'}|, (\Psi_{j\omega'}|$$

[21] Moreover, we can obtain that $\alpha \to 0$, not only that $|\alpha| \to 0$.

the cobasis corresponding to the eigenbasis of the complete diagonalized hamilitonian $H^0 + H^1$, that can be computed using the perturbation method of paper (Laura *et al.*, 1999), e.g. at the first-order it reads

$$(\Psi_{jj'}| = (jj'|, (\Psi_{\omega}| = (\omega|, (\Psi_{\omega\omega'}| = (jj| - (\omega = \Omega_{\mathrm{j}}|, (\Psi_{\omega j'}|$$
$$= (\omega j'|, (\Psi_{j\omega'}| = (j\omega'|$$

Then, the quantum state in the last basis, reads:

$$(\rho(t)| = \sum_{jj'} \rho_{jj'}(\Psi_{jj'}|e^{i\lambda_{jj'}t} + \int_{\Gamma} d\omega\rho_{\omega}(\Psi_{\omega}| + \int_{\overline{\Gamma}}\int_{\Gamma} \rho_{\omega\omega'}(\Psi_{\omega\omega'}|e^{i\lambda_{\omega\omega'}t}$$
$$+ \sum_{j'} \int_{\overline{\Gamma}} d\omega\rho_{\omega j'}(\Psi_{\omega j'}|e^{i\lambda_{\omega j'}t} + \sum_{\mathrm{j}} \int_{\Gamma} d\omega' \rho_{j\omega'}(\Psi_{j\omega'}|$$

where we have used the same integration curves on the complex plane as in section 3.2 of Laura *et al.* (1990). The λ's are the eigenvectors of the complete hamiltonian, that can be computed by the perturbative method, as explained.

Now, it is clear that the decaying times of the discrete terms are given by the imaginary part of the coordinates of the matrix $\lambda_{jj'}$ of Laura *et al.* (1990). Following the method of the quoted paper, it turns out that, at second order of the corresponding perturvative expansion, the imaginary part of $\lambda_{jj'}$ with $j \neq j'$ reads

$$\pi(V_{\Omega_{\mathrm{j}} j} V_{j\Omega_{\mathrm{j}}} + V_{\Omega_{j'} j'} V_{j'\Omega_{j'}})$$

and the inverse of this quantities are the decaying times of the discrete off-diagonal terms, and the biggest of them is the decoherence time. The discrete-continuous decaying times can be also computed, but generally they are not important since the continuous spectrum usually is the environment, and it is also usually traced away.

In this way we can obtain the decoherence time, which is finite for most important interacting systems. Recurrence time is always infinite in system with continuous evolution spectra, like those we are studying (see Gaioli and Garcia Alvarez (in press) for a study of recurrence time for discrete and continuous spectra). So, *decoherence time is always smaller than recurrence time in our model*.

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